THE BERRY PHASE FOR SIMPLE HARMONIC OSCILLATORS

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Abstract. We evaluate the Berry phase for a "missing" family of the square integrable wavefunctions for the linear harmonic oscillator, which cannot be derived by the separation of variables (in a natural way). Instead, it is obtained by the action of the maximal kinematical invariance group on the standard solutions. A simple closed formula for the phase (in terms of elementary functions) is found by integration with the help of a computer algebra system.

Recent reports on observations of the dynamical Casimir effect [27], [51] strengthens the interest to 'nonclassical' states in quantum optics and generalized harmonic oscillators [10], [11], [13], [14], [15], [17], [33], [34] and [37]. The amplification of quantum fluctuations by modulating parameters of an oscillator is closely related to the process of particle production in quantum fields [11], [24], [34] and [37]. Other dynamical amplification mechanisms include the Unruh effect [47] and Hawking radiation [20], [21].

The purpose of this paper is to evaluate the Berry phase for certain "missing" solutions of the time-dependent Schrödinger equation for the linear harmonic oscillator as an instructive example. Applications will be discussed elsewhere.

1. Hidden Solutions

The time-dependent Schrödinger equation for the simple harmonic oscillator,

$$2i\psi_t + \psi_{xx} - x^2\psi = 0, \tag{1.1}$$

has the following six-parameter family of (square integrable) solutions [32]:

$$\psi_n(x,t) = \frac{e^{i\left(\alpha(t)x^2+\delta(t)x+\kappa(t)\right)+i(2n+1)\gamma(t)}}{\sqrt{2^n n!\mu(t)\sqrt{\pi}}}\ e^{-(\beta(t)x+\varepsilon(t))^2/2}\ H_n\left(\beta(t)x+\varepsilon(t)\right), \tag{1.2}$$

where $H_n(x)$ are the Hermite polynomials [40] and

$$\mu(t) = \mu_0\sqrt{\beta_0^4\sin^2 t + (2\alpha_0\sin t + \cos t)^2}, \tag{1.3}$$

$$\alpha(t) = \frac{\alpha_0\cos 2t + \sin 2t\ \left(\beta_0^4 + 4\alpha_0^2 - 1\right)/4}{\beta_0^4\sin^2 t + (2\alpha_0\sin t + \cos t)^2}, \tag{1.4}$$

$$\beta(t) = \frac{\beta_0}{\sqrt{\beta_0^4\sin^2 t + (2\alpha_0\sin t + \cos t)^2}}, \tag{1.5}$$

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$$\gamma(t) = \gamma_0 - \frac{1}{2}\arctan\frac{\beta_0^2\sin t}{2\alpha_0\sin t+\cos t}, \tag{1.6}$$

$$\delta(t) = \frac{\delta_0(2\alpha_0\sin t+\cos t)+\varepsilon_0\beta_0^3\sin t}{\beta_0^4\sin^2 t+(2\alpha_0\sin t+\cos t)^2}, \tag{1.7}$$

$$\varepsilon(t) = \frac{\varepsilon_0(2\alpha_0\sin t+\cos t)-\beta_0\delta_0\sin t}{\sqrt{\beta_0^4\sin^2 t+(2\alpha_0\sin t+\cos t)^2}}, \tag{1.8}$$

$$\begin{aligned}\kappa(t) &= \kappa_0+\sin^2 t\,\frac{\varepsilon_0\beta_0^2(\alpha_0\varepsilon_0-\beta_0\delta_0)-\alpha_0\delta_0^2}{\beta_0^4\sin^2 t+(2\alpha_0\sin t+\cos t)^2}\\ &\quad+\frac{1}{4}\sin 2t\,\frac{\varepsilon_0^2\beta_0^2-\delta_0^2}{\beta_0^4\sin^2 t+(2\alpha_0\sin t+\cos t)^2}\end{aligned} \tag{1.9}$$

($\mu_0\neq 0, \alpha_0, \beta_0\neq 0, \gamma_0, \delta_0, \varepsilon_0, \kappa_0$ are real initial data). These solutions have been derived analytically in the framework of a unified approach to generalized harmonic oscillators (see, for example, [8], [9], [29], [52], [53] and the references therein). They are also verified by a direct substitution with the aid of *Mathematica* computer algebra system [26], [31]. (The simplest special case $\mu_0=\beta_0=1$ and $\alpha_0=\gamma_0=\delta_0=\varepsilon_0=\kappa_0=0$ reproduces the textbook solution obtained by the separation of variables [42], [18], [28], [35]. The shape-preserving oscillator evolutions occur when $\alpha_0=0$ and $\beta_0=1$ and a special case when $\alpha_0=0$ is discussed in [22]. More details on the derivation of these formulas and some *Mathematica* animations, revealing a new feature – an oscillation in space of the probability density $|\psi(x,t)|^2$ – of these solutions, can be found in Refs. [26], [30] and [31].)

The "dynamic harmonic oscillator states" (1.2)–(1.9) are eigenfunctions,

$$E(t)\,\psi_n(x,t) = \left(n+\frac{1}{2}\right)\psi_n(x,t), \tag{1.10}$$

of the time-dependent quadratic invariant,

$$E(t) = \frac{1}{2}\left[\frac{(p-2\alpha x-\delta)^2}{\beta^2}+(\beta x+\varepsilon)^2\right], \qquad \frac{d}{dt}\langle E\rangle = 0, \tag{1.11}$$

where $p=i^{-1}\partial/\partial x$ and the required operator identity,

$$\frac{\partial E}{\partial t}+i^{-1}[E,H]=0, \qquad H=\frac{1}{2}\left(p^2+x^2\right), \tag{1.12}$$

holds [41].

The (isomorphic) maximum kinematical invariance groups of the free particle and harmonic oscillator were introduced in [1], [2], [19], [23], [38] and [39] (see also [7], [25], [36], [48] and the references therein). We have established a connection with certain Ermakov-type system which allows us to bypass a complexity of the traditional Lie algebra approach [30], [32]. (A general procedure of obtaining new solutions by acting on any set of given ones by enveloping algebra of generators of the Heisenberg–Weyl group is described in [15]; see also [3], [4] and [14].)

2. Evaluation of the Phase

The holonomic effect in quantum mechanics known as Berry's phase [5], [6], [43], [50] has received considerable attention over the years (see, for example, [16], [49] and the other references in [41]).

The derivative of Berry's phase has been recently calculated for the generalized harmonic oscillators as follows [41]:

$$\frac{d\theta_n}{dt} = -\beta^{-2}\left(\varepsilon^2 + n + \frac{1}{2}\right)\frac{d\alpha}{dt} + \varepsilon\beta^{-1}\frac{d\delta}{dt} - \frac{d\kappa}{dt}, \tag{2.1}$$

where we are going to use (1.4)–(1.9) and simplify. Integrating by parts, one gets

$$\begin{aligned} \theta_n &= -\left(n+\frac{1}{2}\right)\int \beta^{-2}\frac{d\alpha}{dt}\ dt - \left(\frac{\varepsilon}{\beta}\right)^2\alpha + \frac{\varepsilon\delta}{\beta} - \kappa \\ &\quad + \int\left[\alpha\frac{d}{dt}\left(\frac{\varepsilon}{\beta}\right)^2 - \delta\frac{d}{dt}\left(\frac{\varepsilon}{\beta}\right)\right]\ dt. \end{aligned} \tag{2.2}$$

Here,

$$\begin{aligned} 4\beta_0^2\left[\left(\frac{\varepsilon}{\beta}\right)^2\alpha - \frac{\varepsilon\delta}{\beta} + \kappa\right] &= 2\beta_0\left(2\beta_0\kappa_0 - \delta_0\varepsilon_0\right) \\ &\quad +2\varepsilon_0\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)\cos 2t + \left[\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 - \varepsilon_0^2\right]\sin 2t, \end{aligned} \tag{2.3}$$

$$\begin{aligned} 4\beta_0^2\int\left[\alpha\frac{d}{dt}\left(\frac{\varepsilon}{\beta}\right)^2 - \delta\frac{d}{dt}\left(\frac{\varepsilon}{\beta}\right)\right]\ dt &= 2t\left[\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2\right] \\ &\quad +2\varepsilon_0\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)\cos 2t + \left[\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 - \varepsilon_0^2\right]\sin 2t, \end{aligned} \tag{2.4}$$

$$\int \beta^{-2}\frac{d\alpha}{dt}\ dt = -t\frac{4\alpha_0^2 + \beta_0^4 + 1}{2\beta_0^2} + \arctan\left[\frac{2\alpha_0 + \left(4\alpha_0^2 + \beta_0^4\right)\tan t}{\beta_0^2}\right] \tag{2.5}$$

with the aid of *Mathematica* (the notebook is available from the author's website [45]).

Finally, we evaluate Berry's phase in a closed form:

$$\begin{aligned} \theta_n(t) &= -\left(n+\frac{1}{2}\right)\left[\arctan\left(\frac{2\alpha_0 + \left(4\alpha_0^2 + \beta_0^4\right)\tan t}{\beta_0^2}\right) - \arctan\left(\frac{2\alpha_0}{\beta_0^2}\right) - t\frac{4\alpha_0^2 + \beta_0^4 + 1}{2\beta_0^2}\right] \\ &\quad + t\frac{\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2}{2\beta_0^2}, \qquad \theta_n(0) = 0. \end{aligned} \tag{2.6}$$

(This expression has been verified by differentiation with the help of *Mathematica* once again [45]. Examples are presented in Figure 1.) To the best of our knowledge, this formula is also missing in the available literature — in the simplest case $\beta_0 = 1$ and $\alpha_0 = \gamma_0 = \delta_0 = \varepsilon_0 = \kappa_0 = 0$, one obtains $\theta_n = 0$, which is a well-known result for the textbook solutions. Our formula implies that for the shape-preserving oscillator evolutions, when $\alpha_0 = 0$ and $\beta_0 = 1$, the phase does not depend on n.

On the second hand, Eq. (42) of Ref. [41] gives an alternative formula for evaluation of the phase,

$$\theta_n = (2n+1)\gamma + \int \langle H\rangle\ dt, \tag{2.7}$$

where

$$\langle H\rangle = \frac{1}{2}\left[\langle p^2\rangle + \langle x^2\rangle\right] = \left(n+\frac{1}{2}\right)\frac{1+4\alpha_0^2+\beta_0^4}{2\beta_0^2} + \frac{\left(2\alpha_0\varepsilon_0 - \beta_0\delta_0\right)^2 + \varepsilon_0^2}{2\beta_0^2} \tag{2.8}$$

by (A.3)–(A.5) of Ref. [32]. As a result one gets

$$\theta_n = -\left(n+\frac{1}{2}\right)\arctan\frac{\beta_0\tan t}{1+2\alpha_0\tan t}+\left(n+\frac{1}{2}\right)\frac{1+4\alpha_0^2+\beta_0^4}{2\beta_0^2}t+\frac{\left(2\alpha_0\varepsilon_0-\beta_0\delta_0\right)^2+\varepsilon_0^2}{2\beta_0^2}t, \tag{2.9}$$

which is equivalent to our previuos expression (2.6) up to an elementary transformation.

3. A Conclusion

In addition to the oscillation in space of the probability density $|\psi(x,t)|^2$, which has already been computer animated in [31] and [32], the "dynamic harmonic states" (1.2)–(1.9) possess the nontrivial Berry phase. These two distinguished features of the quantum motion under consideration might be observed in a clever experiment.

Moreover, the electromagnetic field quantization presents the EM field in nonstationary media as a set of harmonic oscillators [11] and [12]. Thus the Berry phase evaluated in this paper is somehow related to the squeezed states of light which are produced in the process of parametric amplification. (See also Ref. [46] for other possible applications.)

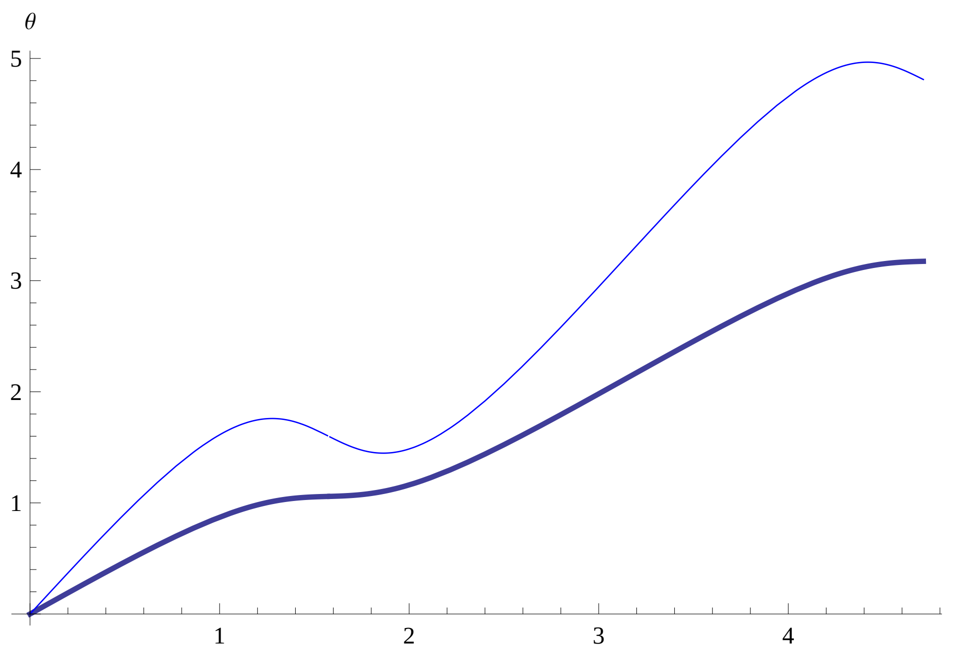

Figure 1. The phases $\theta_0(t)$ and $\theta_1(t)$ with $\alpha_0=\gamma_0=\varepsilon_0=0$, $\beta_0=2/3$ and $\delta_0=1$ (thick and thin lines, respectively).

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